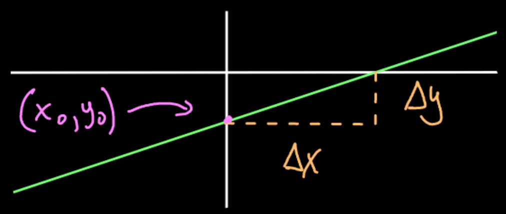
LINES AND PLANES

Recall: In $\mathbb{R}^2$, lines are determined by two pieces of information: ① A point ② A slope.

We think of slope as giving the direction of the line.

We know the slope as $\frac{\Delta y}{\Delta x}$.

This means we can rewrite a line formally as $\ell = \{(x_0 + t\Delta x,\ y_0 + t\Delta y) : t \in \mathbb{R}\}$.



In $\mathbb{R}^3$, this is how we think of lines!

A line is determined by ① a point and ② a direction vector.

Every line can be expressed by ($\vec{p} = \langle x_0, y_0, z_0\rangle$ and $\vec{v} = \langle a, b, c\rangle$):

① $\ell = \{\vec{p} + t\vec{v} : t \in \mathbb{R}\}$ (Set notation)

② $\vec{r}(t) = \vec{p} + t\vec{v}$ (vector notation)

③ $\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$ (parametric equations)

④ $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$ (Symmetric equations)

⇝ NB: If $a = 0$ (for example), we use $x = x_0$ ...

A line segment between two points $A, B \in \mathbb{R}^3$ is the set $\overrightarrow{AB} = \{(1-t)\overrightarrow{OA} + t\,\overrightarrow{OB} : t \in [0,1]\}$.

Ex: Write the line $\ell: \frac{x-3}{5} = \frac{y+1}{2} = \frac{z-6}{6}$ in the other forms.

Sol: We note $x_0 = 3,\ y_0 = -1,\ z_0 = 6,\ a = 5,\ b = 2,\ c = 6$.

Hence $\vec{p} = \langle 3, -1, 6\rangle,\ \vec{v} = \langle 5, 2, 6\rangle$.

Set notation: $\ell = \{\langle 3,-1,6\rangle + t\langle 5,2,6\rangle : t \in \mathbb{R}\}$

Vector Equation: $\vec{r}(t) = \langle 3,-1,6\rangle + t\langle 5,2,6\rangle = \langle 3+5t, -1+2t, 6+6t\rangle$

parametric Equations: $\begin{cases} x = 3+5t \\ y = -1+2t \\ z = 6+6t \end{cases}$

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Ex: Do the given lines intersect?

ⓐ $\vec{r}_1(t) = \langle 1, -2, 4 \rangle + t\langle 1, 3, -1 \rangle$, $\vec{r}_2(t) = \langle 0, 3, -3 \rangle + t\langle 2, 1, 4 \rangle$.

Sol: $\vec{r}_1(t) = \langle 1+t, -2+3t, 4-t \rangle$, $\vec{r}_2(t) = \langle 2t, 3+t, -3+4t \rangle$

To see if $\vec{r}_1(t)$ intersects $\vec{r}_2(t)$, we'll solve (attempt!)

$\boxed{\vec{r}_1(t) = \vec{r}_2(s)} \iff \langle 1+t, -2+3t, 4-t \rangle = \langle 2s, 3+s, -3+4s \rangle$

$$\iff \begin{cases} 1+t = 2s \\ 3t-2 = s+3 \\ 4-t = 4s-3 \end{cases} \iff \begin{cases} t = 2s-1 \\ 3(2s-1)-2 = s+3 \\ 4-(2s-1) = 4s-3 \end{cases} \iff \begin{cases} t = 2s-1 \\ 6s-3-2 = s+3 \\ 4-2s+1 = 4s-3 \end{cases}$$

$$\iff \begin{cases} t = 11/5 \\ s = 8/5 \\ 6s = 8 \end{cases} \iff \begin{cases} t = 11/5 \\ s = 8/5 \\ s = 8/6 \end{cases} \neq$$

$\therefore$ There is no intersection point! ////

ⓑ $\vec{r}_1(t) = \langle -2, -1, 3 \rangle + t\langle 1, 1, 1 \rangle$, $\vec{r}_2(t) = \langle 0, -1, 1 \rangle + t\langle 1, 0, -1 \rangle$

Sol: $\vec{r}_1(t) = \langle t-2, t-1, t+3 \rangle$, $\vec{r}_2(t) = \langle t, -1, 1-t \rangle$

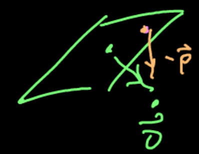
$\vec{r}_1(s) = \vec{r}_2(t) \iff \langle s-2, s-1, s+3 \rangle = \langle t, -1, 1-t \rangle$

$$\iff \begin{cases} s-2 = t \\ s-1 = -1 \\ s+3 = 1-t \end{cases} \iff \begin{cases} t = -2 \\ s = 0 \\ 3 = 1-(-2) \end{cases} \iff \begin{cases} s = 0 \\ t = -2 \\ 3 = 3 \leftarrow \text{True!} \end{cases}$$

$$\iff \begin{cases} s = 0 \\ t = -2 \end{cases}$$

Hence $\vec{r}_1(t)$ intersects $\vec{r}_2(t)$ at $\vec{r}_1(0) = \langle -2, -1, 3 \rangle$. ////

Planes in $\mathbb{R}^3$ are given by a point (x_0, y_0, z_0) in the plane and a nonzero **normal vector** $\vec{n}$ to which all vectors the (homogeneous) plane are orthogonal.





The plane P through point (x_0, y_0, z_0) with normal vector $\vec{n} = \langle a, b, c\rangle$ can be expressed as...

① $P = \{ \vec{x} : \vec{n}\cdot(\vec{x}-\vec{p}) = 0 \}$ ← Set (set notation)

② $\vec{n}\cdot\langle x - x_0, y - y_0, z - z_0\rangle = 0$ ← equations (vector equation)

③ $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$ (scalar equation)

Ex: Compute vector and scalar equations for the plane...

ⓐ through point $(-1, \frac{1}{2}, 3)$ with normal vector $\langle 1,4,1\rangle$.

Sol: $\vec{n} = \langle 1,4,1\rangle$, $\vec{p} = \langle -1, \frac{1}{2}, 3\rangle$

Vector Equation: $\vec{n}\cdot(\vec{x}-\vec{p}) = 0$ i.e. $\langle 1,4,1\rangle \cdot \langle x-(-1), y-\frac{1}{2}, z-3\rangle = 0$

scalar equation: $1(x-(-1)) + 4(y-\frac{1}{2}) + 1(z-3) = 0$

$\iff x + 4y + z = 4$ ☑

ⓑ through the points $\underset{P}{(2,1,2)}$, $\underset{Q}{(3,-8,6)}$, $\underset{R}{(-2,-3,1)}$.

Sol: $\vec{PQ} = \langle 3-2, -8-1, 6-2\rangle = \langle 1,-9,4\rangle$, $\vec{QR} = \langle -2-2, -3-1, 1-2\rangle = \langle -4,-4,-1\rangle$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -9 & 4 \\ -4 & -4 & -1 \end{vmatrix} = \left\langle \begin{vmatrix} -9 & 4 \\ -4 & -1 \end{vmatrix}, -\begin{vmatrix} 1 & 4 \\ -4 & -1 \end{vmatrix}, \begin{vmatrix} 1 & -9 \\ -4 & -4 \end{vmatrix} \right\rangle = \langle 25, -15, -40\rangle$$

Vec EQ: $\vec{n}\cdot(\vec{x}-\vec{p}) = 0$ i.e. $\langle 25,-15,-40\rangle \cdot \langle x+2, y+3, z-1\rangle = 0$

Scalar Eq: $25(x+2) - 15(y+3) - 40(z-1) = 0$

or: $5(x+2) - 3(y+3) - 8(z-1) = 0$ ☑

Ex: Compute plane through the point $(5,3,5)$ and parallel to the plane $x+y+z=3$.

Sol: $\vec{n} = \langle 1,1,1\rangle \leftarrow$ parallel to $x+y+z=3$.

$\vec{p} = \langle 5,3,5\rangle$

$\vec{n}\cdot(\vec{x}-\vec{p}) = 0$

$\therefore$ the desired plane has equation $\langle 1,1,1\rangle \cdot \langle x-5, y-3, z-5\rangle = 0$

i.e. $(x-5)+(y-3)+(z-5)=0$

i.e. $x+y+z=13$ $\blacksquare$

Ex: The plane passing through the point $(3,1,4)$ and containing the line of intersection of the planes

$x+2y+3z=1$ and $2x-y+z=-3$.

Sol: First: Compute the line.

$$\begin{cases} x+2y+3z=1 \\ 2x-y+z=-3 \end{cases} \iff \begin{cases} 2x+4y+6z=2 \\ -5y-5z=-5 \end{cases} \iff \begin{cases} x+2y+3z=1 \\ y=1-z \end{cases}$$

$$\iff \begin{cases} x+2(1-z)+3z=1 \\ y=1-z \end{cases} \iff \begin{cases} x+z=-1 \\ y=1-z \end{cases} \iff \begin{cases} x=-1-z \\ y=1-z \end{cases}$$

$\therefore$ the line is $\vec{r}(t) = \langle -1,1,0\rangle + t\langle -1,-1,1\rangle$. Can obtain two more points: $t=0$: $(-1,1,0)$, $t=-1$: $(0,2,-1)$

Two vectors for the plane: $\vec{u} = \langle 3-(-1), 1-1, 4-0\rangle = \langle 4,0,4\rangle$

$\vec{v} = \langle 3-0, 1-2, 4-(-1)\rangle = \langle 3,-1,5\rangle$

$$\vec{n} = \left(\tfrac{1}{4}\vec{u}\right)\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 3 & -1 & 5 \end{vmatrix} = \left\langle \begin{vmatrix} 0 & 1 \\ -1 & 5 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} \right\rangle = \langle 1,-2,-1\rangle$$

$\therefore$ The plane has equation $\vec{n}\cdot(\vec{x}-\vec{p})=0$ i.e. $(x-3)-2(y-1)-(z-4)=0$ $\blacksquare$

Ex: Find the intersection point of the line

$5x = y/2 = z+2$ and the plane $10x - 7y + 3z + 24 = 0$.

Sol: the line has symmetric equations $\frac{x-0}{1/5} = \frac{y-0}{2} = \frac{z-(-2)}{1}$, so it has vector equation

$$\vec{r}(t) = \langle 0, 0, -2 \rangle + t \langle \tfrac{1}{5}, 2, 1 \rangle = \langle \tfrac{1}{5}t, 2t, t-2 \rangle$$

At the intersection we have

$$5(\tfrac{1}{5}t) - 7(2t) + 3(t-2) + 24 = 0$$

$$\Leftrightarrow \quad t - 14t + 3t - 6 + 24 = 0 \quad \Leftrightarrow \quad -10t + 18 = 0 \Leftrightarrow t = \tfrac{9}{5}.$$

Hence the point of intersection is $\left(\frac{9}{25}, \frac{18}{5}, -\frac{1}{5}\right)$, because $\vec{r}(9/5) = \langle \frac{9}{25}, \frac{18}{5}, -\frac{1}{5} \rangle$. ∎

Ex: Are the given planes parallel, orthogonal, or neither?

ⓐ $x + 4y - 3z = 1$ and $-3x + 6y + 7z = 0$

$\vec{n}_1 = \langle 1, 4, -3 \rangle$ $\qquad \vec{n}_2 = \langle -3, 6, 7 \rangle$

$\vec{n}_2 = c\vec{n}_1 \Rightarrow \langle -3, 6, 7 \rangle = \langle c, 4c, -3c \rangle$

$\Rightarrow -3 = c$ and $c = \frac{3}{2}$ ✗ absurd!

∴ these planes are not parallel.

$\vec{n}_1 \cdot \vec{n}_2 = -3 + 24 - 21 = 0$ ∴ The planes are orthogonal!

ⓑ $9x - 3y + 6z = 2$ and $2y = 6x + 4z \rightsquigarrow 0 = 6x - 2y + 4z$

$\vec{n}_1 = \langle 9, -3, 6 \rangle = 3\langle 3, -1, 2 \rangle$ $\qquad \vec{n}_2 = \langle 6, -2, 4 \rangle = 2\langle 3, -1, 2 \rangle$

$\therefore \vec{n}_1 = \frac{3}{2} \cdot 2 \langle 3, -1, 2 \rangle = \frac{3}{2}\vec{n}_2.$

∴ The planes are parallel. ∎